THE APPROXIMATE SUMMATION OF n TERMS OF ANY HARMONIC SERIES.

By O. L. CALLECOT, Omaha, Nebraska.

For any series $U_1+U_2+\ldots\ldots$, we shall give notations to certain groups of successive terms; thus,

$$G_1=U_{m+1}+U_{m+2}+\ldots\ldots+U_{m+x_1},\quad G_2=U_{m+x_1+1}+\ldots\ldots+U_{m+x_1+x_2},\quad \ldots\ldots$$

Then for $n>m$, we have

$$(1)\qquad E\equiv(G_1+\ldots\ldots+G_n)-(U_1+\ldots\ldots+U_n)=(G_1-U_1)+\ldots\ldots+(G_n-U_n),$$

$$(2)\qquad E=(U_{n+1}+U_{n+2}+\ldots\ldots+U_{m+x_1+\ldots+x_n})-(U_1+U_2+\ldots\ldots+U_m),$$

since the terms $U_{m+1}, \ldots\ldots, U_n$ are common to $G_1+\ldots\ldots+G_n$ and $U_1+\ldots\ldots+U_n$.

Applying this method to the special harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$, and taking $m=1$, $x=y=w=\ldots\ldots=z=3$, we have, by (1) and (2),

$$(3)\qquad E=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\ldots+\left(\frac{1}{3n-1}+\frac{1}{3n}+\frac{1}{3n+1}\right).$$

$$-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)=\frac{2}{2.3.4}+\frac{2}{5.6.7}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)},$$

$$(4)\qquad E=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n+1}-1.$$

$$(5)\qquad \text{Hence } \frac{1}{n+1}+\ldots+\frac{1}{3n+1}=1+\frac{2}{2.3.4}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)}.$$

But

$$(6)\qquad \frac{2}{2.3.4}+\frac{2}{5.6.7}+\ldots\infty=.098612\ldots\ldots$$

$$(7)\qquad \frac{2}{2.3.4}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)}=.098612\ldots-\frac{1}{3(3n+1)(3n+2)}+\delta,$$

$$\delta<\frac{2}{(3n+1)(3n+2)(3n+3)(3n+4)}.$$

Now any number which gives the remainder 1 on division by 3 may be taken as the last in a group of $2n+1$ terms. We may therefore arrange the series $1+\frac{1}{2}+\ldots\ldots$ in successive groups of $2n+1$ terms by throwing out 0, 1, or 2 terms for each group, as the case may be.

(8) Hence $1+\frac{1}{2}+\ldots+\frac{1}{n'}=G+D+U$,

where G is the number of groups in n' terms, U is the sum of the deleted terms, and D is the sum of the difference groups, $\frac{2}{2.3.4}+\ldots+\frac{2}{(3n-1)(3n)(3n+1)}$. If n is not taken smaller than 15,

$$(9) \qquad D=(.098612\ldots\ldots)G-\frac{10^G-1}{27(10^{G-1})\ (3g+1)\ (3g+2)}-\delta,$$

$$\delta<\frac{1}{216(3g+1)(3g+2)},$$

where g is the n of the group of lowest denominator.

Since the formula applies to the series $\frac{1}{r}+\frac{1}{r+1}+\ldots\ldots$, $r>15$, we apply the following table for the sum of the earlier terms.

TABLE GIVING SUM OF FIRST n TERMS OF SERIES $1+\frac{1}{2}+\ldots\ldots$

n	*Sum*	n	*Sum*
2	1.5	23	3.734291
3	1.833333	24	3.775958
4	2.083333	25	3.815958
5	2.283333	26	3.854419
6	2.449999	27	3.891456
7	2.592857	28	3.927171
8	2.717857	29	3.961653
9	2.828968	30	3.994987
10	2.928968	31	4.027245
11	3.019877	32	4.058495
12	3.103210	33	4.088798
13	3.180133	34	4.110209
14	3.251562	35	4.138781
15	3.318228	36	4.166559
16	3.380728	37	4.193586
17	3.439552	38	4.219901
18	3.495108	39	4.245542
19	3.547739	40	4.270542
20	3.597739	41	4.294933
21	3.645338	42	4.318742
22	3.690813	43	4.341998

Example. Find the sum approximately of $1+\frac{1}{2}+\ldots+\frac{1}{10{,}000}$.

	Denominators of terms in U.	
10000		
3) 9999	0	0
3333	3333	3332
3331		
3) 3330		
1110	1110	1109
1108		
3) 1107		
369	369	368
367		
3) 366		
122	122	0
121		
3) 120; ——40		

$B=4.270542$

$U=0.016088$

$(1.098612)5=5.493061$

9.779692

Second term of (9) $=-0.000025$

Sum $=9.77966$ ..... $-\delta$, $\delta<.000{,}000{,}313$.

Any harmonic series $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots$ may be reduced to the form

$$\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\frac{1}{n+2+f}+\ldots\ldots\right)$$

where f is the fractional part of the result of dividing a by d, and n is the integral part of the result. Also,

$$\frac{1}{d}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{n'}\right)-\frac{1}{d}\left(\frac{1}{n+f}+\frac{1}{n+1+f}+\ldots+\frac{1}{n'+f}\right)$$

$$=\frac{1}{d}\left(\frac{f}{n(n+f)}+\frac{f}{(n+1)(n+1+f)}+\ldots+\frac{f}{n'(n'+f)}\right).$$

Since the value of $\frac{1}{d}\left(\frac{1}{n}+\ldots+\frac{1}{n'}\right)$ may be found according to methods already

given, and since the series $\frac{1}{d}\left(\frac{f}{n(n+f)}+\ldots\ldots\right)$ converges rapidly, we may approximately determine the sum of n terms of any harmonic series.

NOTE ON THE MAXIMUM INDICATOR OF CERTAIN ODD NUMBERS.

By REV. R. D. CARMICHAEL, Hartselle, Alabama.

If p, the least prime factor of N, is of the form $4l+1$, the maximum indicator of N is a multiple of 4.

In the MONTHLY, May, 1905, p. 107, I have shown that, for p a prime of the form $4l+1$, we have

$$\left(1.2.3\ldots\ldots\frac{p-1}{2}\right)^2\equiv -1 \pmod{p}.$$

Hence, if $\left(1.2.3\ldots\ldots\frac{p-1}{2}\right)^{2n}\equiv 1 \pmod{N}$, n is even; and therefore the maximum indicator of N, being a multiple of the least value $2n$ satisfying the above congruence, is also a multiple of 4.

Corollary. *The equation $y^4=mx+1$ has at least one positive integral solution when the least factor of m is congruent to unity modulo 4.*

By Wilson's theorem, it is easily shown as above that

The maximum indicator of any odd number is even.

Corollary. *The equation $y^2=mx+1$ has at least one positive integral solution, when m is odd.*

If p and $2p-1$ are odd primes, the maximum indicator of $p(2p-1)$ is a multiple of 4.

As in the first congruence above we have

$$(1.2.3\ldots\ldots\overline{p-1})^2\equiv -1 \pmod{2p-1}.$$

By Wilson's theorem,

$$(1.2.3\ldots\ldots\overline{p-1})^2\equiv 1 \pmod{p}.$$

$$\text{Thus } (1.2.3\ldots\ldots\overline{p-1})^4\equiv 1 \pmod{p.\overline{2p-1}}.$$

Now, $(1.2.3\ldots\ldots\overline{p-1})$ is not congruent to 1 modulo $p.\overline{2p-1}$. It is thus shown that one indicator, at least, is 4. Hence the maximum indicator is a multiple of 4.

ON A FUNDAMENTAL THEOREM IN TRIGONOMETRY.

By G. A. MILLER.

Many of our elementary text-books on trigonometry give the values of the trigonometric functions of the following eight angles: $\pm x$, $90 \pm x$, $180 \pm x$, $270 \pm x$. It has been observed that these eight angles may be derived from any one of them by means of the operations of a well known group which is known as the octic group, or the group of the square.* The present note is intended to exhibit more clearly the value of this point of view by presenting the matter in a somewhat different form from the one given in the article to which reference has been made.

We shall represent by c the operations of taking the complement, while p will be used to represent that the angle is increased by $90°$. As angles which differ as a multiple of $360°$ are considered identical these operations are of periods 2 and 4, respectively. In fact the first is of period 2 independently of the modulus. The operations of increasing an angle by $180°$ and by $270°$ will be denoted bp p^2 and p^3, respectively, since these results are obtained by increasing the angle twice or three times by $90°$. In accord with this $p^5=p$, $c^3=c$, $p^4=c^2=1$.

The eight operations of the octic group may be written as follows:

$$\begin{array}{cccc} 1 & p & p^2 & p^3 \\ c & cp & cp^2 & cp^3 \end{array}$$

If these operations are performed on the angle x the resulting angles are respectively:

$$\begin{array}{cccc} x & x+90° & x+180° & x+270° \\ 90°-x & 180°-x & 270°-x & -x \end{array}$$

By means of these results the functions of the eight angles mentioned above may be expressed in terms of functions of x provided we know the functions of $cx=90-x$, and of $px=x+90°$. The former functions are simply a matter of definition while the latter are given in the following table:

$$\begin{aligned} &\sin px=\sin(90°+x)=\cos x \\ &\cos px=\cos(90°+x)=-\sin x \\ &\tan px=\tan(90°+x)=-\cot x \\ &\cot px=\cot(90°+x)=-\tan x \\ &\sec px=\sec(90°+x)=-\csc x \\ &\csc px=\csc(90°+x)=\sec x. \end{aligned}$$

Quarterly Journal of Mathematics, Vol. 37 (1906), p. 226.

Hence,

$$\sin p^2x=\sin(180^\circ+x)=\cos px=-\sin x^*$$
$$\cos p^2x=\cos(180^\circ+x)=-\sin px=-\cos x$$
$$\tan p^2x=\tan(180^\circ+x)=-\cot px=\tan x$$
$$\cot p^2x=\cot(180^\circ+x)=-\tan px=\cot x$$
$$\sec p^2x=\sec(180^\circ+x)=-\csc px=-\sec x$$
$$\csc p^2x=\csc(180^\circ+x)=\sec px=-\csc x.$$

$$\sin p^3x=\sin(270^\circ+x)=\cos p^2x=-\cos x\dagger$$
$$\cos p^3x=\cos(270^\circ+x)=-\sin p^2x=\sin x$$
$$\tan p^3x=\tan(270^\circ+x)=-\cot p^2x=-\cot x$$
$$\cot p^3x=\cot(270^\circ+x)=-\tan p^2x=-\tan x$$
$$\sec p^3x=\sec(270^\circ+x)=-\csc p^2x=\csc x$$
$$\csc p^3x=\csc(270^\circ+x)=\sec p^2x=-\sec x.$$

$$\sin cpx=\sin(180^\circ-x)=\cos cx=\sin x$$
$$\cos cpx=\cos(180^\circ-x)=-\sin cx=-\cos x$$
$$\tan cpx=\tan(180^\circ-x)=-\cot cx=-\tan x$$
$$\cot cpx=\cot(180^\circ-x)=-\tan cx=-\cot x$$
$$\sec cpx=\sec(180^\circ-x)=-\csc cx=-\sec x$$
$$\csc cpx=\csc(180^\circ-x)=\sec cx=\csc x.$$

$$\sin cp^2x=\sin(270^\circ-x)=-\sin cx=-\cos x\ddagger$$
$$\cos cp^2x=\cos(270^\circ-x)=-\cos cx=-\sin x$$
$$\tan cp^2x=\tan(270^\circ-x)=\tan cx=\cot x$$
$$\cot cp^2x=\cot(270^\circ-x)=\cot cx=\tan x$$
$$\sec cp^2x=\sec(270^\circ-x)=-\sec cx=-\csc x$$
$$\csc cp^2x=\csc(270^\circ-x)=-\csc cx=-\sec x.$$

$$\sin cp^3x=\sin-x=-\cos cx=-\sin x$$
$$\cos cp^3x=\cos-x=\sin cx=\cos x$$
$$\tan cp^3x=\tan-x=-\cot cx=-\tan x$$
$$\cot cp^3x=\cot-x=-\tan cx=-\cot x$$
$$\sec cp^3x=\sec-x=\csc cx=\sec x$$
$$\csc cp^3x=\csc-x=-\sec cx=-\csc x$$

The advantages of the present view point result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group we know directly what two angles may be made fundamental in the study of the others. The necessary and sufficient condition is that the two angles correspond to generators in

*Sin $p^3x=\sin px=\cos z$, where $z=px$.
†Sin $p^3x=\sin py=\cos y$, where $y=p^2x$; also, sin $p^3x=\sin p^2w=-\sin w$, where $w=px$
‡Also, sin $cp^2x=\sin py=\cos y$, where $y=cpx$, etc.

the octic group. In other words they must correspond to two non-commutative operators of this group since any two non-commutative operators of a non-abelian group of order p^3 generate the group and any two generators are non-commutative.

The properties of the octic group suggest an almost endless number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them can be expressed in terms of the functions of any other one of them just as easily as in terms of functions of x. Since the octic group has only one invariant operator besides the identity the corresponding angle cannot be used as one of two fundamental angles, while all the other angles besides the one corresponding to the identity can be used as one of these two angles. If x is the angle corresponding to the identity $180^\circ + x$ corresponds to the invariant operator of order 2. If this angle corresponds to the identity while c and p have the same meaning as before the eight operators of the octic group in the third paragraph correspond respectively to the following angles:

$180^\circ + x$	$270^\circ + x$	x	$90^\circ + x$
$270^\circ - x$	$-x$	$90^\circ - x$	$180^\circ - x$

Hence, for example,

$$\sin p^2(180^\circ+x)=\sin x=\cos p(180^\circ+x)=-\sin(180^\circ+x)$$
$$\cos p^2(180^\circ+x)=\cos x=-\sin p(180^\circ+x)=-\cos(180^\circ+x)$$
$$\tan p^2(180^\circ+x)=\tan x=-\cot p(180^\circ+x)=\tan(180^\circ+x)$$
$$\cot p^2(180^\circ+x)=\cot x=-\tan p(180^\circ+x)=\cot(180^\circ+x)$$
$$\sec p^2(180^\circ+x)=\sec x=-\csc p(180^\circ+x)=-\sec(180^\circ+x)$$
$$\csc p^2(180^\circ+x)=\csc x=\sec p(180^\circ+x)=-\csc(180^\circ+x).$$

The use of this octic group makes the work entirely analytic after the functions of the fundamental angles are known. The main interest however lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this view point even if he did not consider it wise to present it to the beginner.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

259. Proposed by ARTEMUS MARTIN, M. A., Ph. D., LL. D., Washington, D. C.

On page 167 of George Bruce Halsted's *Metrical Geometry* (Mensuration), Boston, 1881, Table of Scalene Triangles, is found the following triangle, viz., Sides 21, 61, 65; Area 420. The sides of a rational scalene triangle, whose sides have no common divisor, can not all be odd; one must be even and the other two odd. It is required to find the error in the sides of the above triangle, assuming that the area is correct.

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let a, b, and x be the three sides of the triangle, and m the area, and we have:

$$\tfrac{1}{2}(a+b+x)\times\tfrac{1}{2}(-a+b+x)\times\tfrac{1}{2}(a-b+x)\times\tfrac{1}{2}(a+b-x)=m^2\text{; or}$$

$$[(a+b)^2-x^2][x^2-(a-b)^2]=(4m)^2\text{;}$$

$$x^4-2(a^2+b^2)x^2+(a^2+b^2)^2=4a^2b^2-(4m)^2\text{;}$$

$$\text{hence, } x=\pm\sqrt{\{(a^2+b^2)\pm 2\sqrt{[(ab+2m)(ab-2m)]}\}}.$$

As x must be a whole number the radical term must be an exact square, which requires the quantity, $(ab+2m)(ab-2m)$ to be an exact square. Now a and b may be: $a=21$, and $b=61$; or $a=21$, and $b=65$; or $a=61$, and $b=65$.

With $a=61$, and $b=65$, but with none of the other values, the inner radical is an exact square. Reducing the outer radical, we have $x=14$, when $m=420$, if we take the lower sign for the inner radical.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The conditions of the problem require that

$$\tfrac{1}{2}(x+y+z)\tfrac{1}{2}(x+y-z)\tfrac{1}{2}(x-y+z)\tfrac{1}{2}(-x+y+z)=2\times2\times2\times2\times3\times3\times5\times5\times7\times7,$$

$$\text{and } x+y+z=(x+y-z)+(x-y+z)+(-x+y+z).$$

Hence we may have,

$$\left.\begin{array}{rrrl} x+y+z= & 120, & 140, & 150, \ldots\ldots \\ x+y-z= & 70, & 112, & 98, \ldots\ldots \\ x-y+z= & 42, & 18, & 48, \ldots\ldots \\ -x+y+z= & 8, & 10, & 4, \ldots\ldots \end{array}\right\} \quad \begin{array}{l} x=56,\ 65,\ 73, \ldots\ldots \\ y=39,\ 61,\ 51, \ldots\ldots \\ z=25,\ 14,\ 26, \ldots\ldots \end{array}$$

Hence, the side given as 21 should be 14.

Also solved in part by the following: S. A. Corey, Henry Heaton, A. H. Holmes, L. E. Newcomb, and J. E. Sanders.

260. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The necessary and sufficient condition that a binary form be a perfect nth power is that its Hessian vanish.

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting $\frac{\partial u}{\partial x}$ by p, $\frac{\partial u}{\partial y}$ by q, the vanishing of the Hessian shows that $p=f(q)$, *i. e.*, $q=mp$, since both p and q are homogeneous and of the same degree. By Lagrange's method of solving partial differential equations, we have

$$\frac{dx}{m}=\frac{dy}{-1}=\frac{du}{0}.$$

Hence, u=constant, $x+my$=constant, and a general solution is given by

$$u=f(x+my)=(x+my)^n,$$

since u is homogeneous in x, y. It is easily verified that when $u=(x+my)^n$ the Hessian vanishes. Hence this condition is both necessary and sufficient.

II. Solution by the PROPOSER.

A slightly different point of view from the above is afforded by the following method:

The Hessian is the Jacobian of the first derivatives p and q. Hence $p-mq=0$. Also $xp+yq=nu$, n being the order of u. Solving for p and q,

$$p=\frac{nmu}{y+mx},\quad q=\frac{nu}{y+mx}.$$

Also, $du=pdx+qdy=nu\frac{dy+mdx}{y+mx}$, or $\frac{du}{u}=n\frac{d(y+mx)}{y+mx}$.

Henge, $\log u=n\log k(y+mx)$, $u=(a_1x+a_2y)^n$.

261. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series, $\frac{1}{n^p}+\frac{3}{n^{2p}}+\frac{5}{n^{3p}}+\frac{7}{n^{4p}}+\frac{9}{n^{5p}}+\ldots\ldots$

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting n^{-p} by x, we have

$$\sum_{i=1}^{\infty}\frac{(2i-1)}{n^{ip}}=x[1+3x+5x^2+7x^3+\ldots\ldots]$$
$$=x\sum(2r+1)x^r=2x\sum rx^r+x\sum x^r$$

$$=2x^2(1-x)^{-2}+x(1-x)^{-1}=x(1+x)(1-x)^{-2}=\frac{n^p+1}{(n^p-1)^2}$$

where we must have $|x|<1$.

Also solved by Henry Heaton, A. H. Holmes, and G. B. M. Zerr.

CALCULUS.

217. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\underset{n=\infty}{\text{Lim}} \frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots(2n)]}$.

I. Solution by the PROPOSER.

Let $x=\underset{n=\infty}{\text{Lim}} \frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots 2n]}$

$$=\underset{n=\infty}{\text{Lim}} \sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\ldots\left(1+\frac{n}{n}\right)}.$$

Then $\log x=\underset{n=\infty}{\text{Lim}} \frac{1}{n}\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\ldots\left(1+\frac{n}{n}\right)\right]$

$$=\underset{n=\infty}{\text{Lim}} \frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\log\left(1+\frac{\lambda}{n}\right)=\underset{n=\infty}{\text{Lim}} \frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\left(\frac{\lambda}{n}-\frac{\lambda^2}{2n^2}+\frac{\lambda^3}{3n^3}-\ldots\ldots\right)$$

$$=\underset{n=\infty}{\text{Lim}} \sum_{\lambda=1}^{\lambda=n}\sum_{\kappa=1}^{\kappa=n}(-1)^{\kappa-1}\frac{\lambda^\kappa}{\kappa n^{\kappa+1}}.$$

If the method of differences is used for $\sum_{\lambda=1}^{\lambda=n}\lambda^\kappa = 1^\kappa + 2^\kappa + 3^\kappa + \ldots\ldots$, the κth series of differences is

$$(\kappa+1)^\kappa-\binom{\kappa}{1}\kappa^\kappa+\binom{\kappa}{2}(\kappa-1)^\kappa-\binom{\kappa}{3}(\kappa-2)^\kappa+\ldots\ldots$$

$$+(-1)^{\kappa-1}\binom{\kappa}{\kappa-1}2^\kappa+(-1)^\kappa 1^\kappa\equiv\kappa!.$$

The $(\kappa+1)$th series is

$$(\kappa+2)^\kappa-\binom{\kappa+1}{1}(\kappa+1)^\kappa+\binom{\kappa+1}{2}\kappa^\kappa-\ldots\ldots+(-1)^\kappa\binom{\kappa+1}{\kappa}2^\kappa+(-1)^{\kappa+1}1^\kappa\equiv 0,$$

κ being a positive integer.

If the first given number is represented by a and the successive differences by $d_1, d_2, \ldots\ldots$

$$S_{n,\kappa}=\binom{n}{1}a+\binom{n}{2}d_1+\binom{n}{3}d_2+\ldots\ldots+\binom{n}{\kappa+1}d_\kappa.$$

First $\underset{n \doteq \infty}{\text{Lim}} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa}$ must be sought.

Only in the last term of $S_{n,\kappa}$ n appears in $\kappa+1$ factors, therefore the preceding terms disappear, and

$$\underset{n \doteq \infty}{\text{Lim}} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa} = \underset{n \doteq \infty}{\text{Lim}} \frac{1}{\kappa n^{\kappa+1}} \left[\frac{n(n-1)(n-2)\ldots\ldots(n-\kappa)}{(\kappa+1)!} \kappa! \right] = \frac{1}{\kappa(\kappa+1)}.$$

Therefore $\underset{n \doteq \infty}{\text{Lim}} \sum_{\lambda=1}^{\lambda=n} \sum_{\kappa=1}^{\kappa=n} (-1)^{\kappa-1} \frac{\lambda^{\kappa}}{\kappa n^{\kappa+1}} = \sum_{\kappa=1}^{\kappa=\infty} (-1)^{\kappa-1} \frac{1}{\kappa(\kappa+1)}$

$$= \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \ldots\ldots = (1 - \frac{1}{2}) - (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) - \ldots\ldots$$

$$= 2(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots) - 1 = 2 \log 2 - 1 = \log \frac{4}{e}.$$

Hence, $\log x = \log \frac{4}{e}$; and $x = \underset{n \doteq \infty}{\text{Lim}} \frac{1}{n} \sqrt[n]{[(n+1)(n+2) \ldots\ldots 2n]} = \frac{4}{e}$.

II. Solution by S. A. COREY, Hiteman, Iowa.

Evidently, $\frac{1}{n} \sqrt[n]{[(n+1)(n+2) \ldots\ldots (2n)]}$

$$= \frac{1}{n} \sqrt[n]{n^n (1 + \frac{1}{n})(1 + \frac{2}{n}) \ldots\ldots (2)}$$

$$= \sqrt[n]{(1 + \frac{1}{n})(1 + \frac{2}{n}) \ldots\ldots (2)} = s \text{ (say)}.$$

Therefore, $\log s = \frac{1}{n} [\log (1 + \frac{1}{n}) + \log (1 + \frac{2}{n}) + \ldots\ldots + \log 2]$.

Letting $dx = 1/n$, we have,

$$\underset{n \doteq \infty}{\text{Lim}} \log s = \int_1^2 \log x \, dx = 2 \log 2 - 1, \text{ or } s = \frac{4}{e}.$$

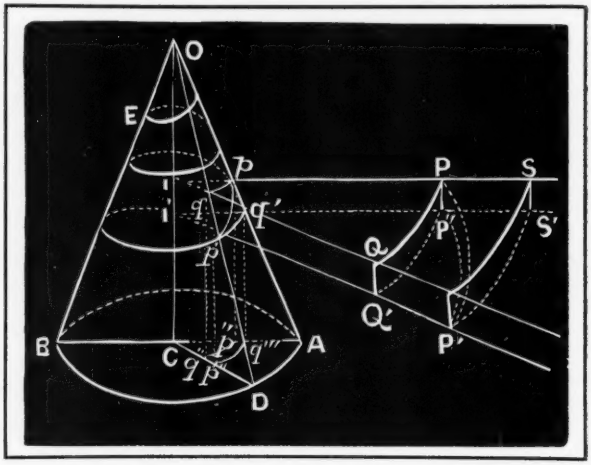
Also solved by Henry Heaton, and J. Scheffer. Several incorrect solutions were received.

209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

Solution by Professor B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Let P and P' be two consecutive positions of the bird at any time; p and p' the two corresponding positions of the end of the string in contact with the cone, it being assumed that the string adheres slightly to the cone in order that the conditions of the problem be fulfilled; $s=pP=OEP$, the length of the string unwound at any time; $Op=w$, $pq'=dw$; $pp'=ds$; $\theta=$ the angle between Ip and the line perpendicular to OC at the beginning of the flight, the angle being measured in the direction in which the bird flys around the cone; $d\theta=$the angle $pIq=$the angle ACD; $s_1=$the distance the bird has flown at any time; $R=$the radius of the base of the cone; and $l=$its slant height. Then



$$ds=[(pq')^2+(q'p')^2]^{\frac{1}{2}}=[(\frac{Rw}{l}d\theta)^2+dw^2]^{\frac{1}{2}} \quad \text{.....(1).}$$

Since the string passes around the cone n times, it follows that

$$w : R\theta=\frac{l}{n} : 2\pi R, \text{ or } w\frac{l\theta}{2\pi n}, \text{ and } dw=\frac{ld\theta}{2\pi n} \quad \text{.....(2).}$$

Hence, $ds=\frac{1}{2\pi n}[R^2\theta^2+l^2]d\theta$.....(3), and

$$s=\frac{1}{2\pi n}\int_0^{\theta}\sqrt{[R^2\theta^2+l^2]}d\theta=\frac{R}{4\pi n}[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})] \quad \text{.....(4),}$$

where $k=l/R$. If $\theta=2\pi n$, this expression gives the complete length of the string.

Now, $ds_1=PP'=[P'Q'^2+Q'P''^2+QQ'^2]^{\frac{1}{2}}$ .....(5).

But $P''Q'=PQ=PId\theta=(Pp+pI)d\theta=(s+\frac{R\theta}{2\pi n})d\theta$; $Q'P'=pp'=ds$

$$= \frac{R}{2\pi n}\sqrt{(k^2+\theta^2)}\,d\theta; \text{ and } QQ' = II' = pq'\cos\angle AOC = \frac{\sqrt{(l^2-R^2)}}{l}\,d\omega$$

$$= \frac{1}{2\pi n}\sqrt{(l^2-R^2)}\,d\theta.$$

Hence, by substituting these values of $P'Q'$, $Q'P''$, and QQ' and the value of s from (4) in (5), we have

$$ds_1 = \frac{R}{4\pi n}\{[\theta\sqrt{(k^2+\theta^2)} + k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k}) + 2\theta]^2 + 4\theta^4 + 8k^2 - 4\}^{\frac{1}{2}}\,d\theta.$$

Hence, $s_1 = \frac{R}{4\pi n}\int_0^{2\pi n}\{[\theta\sqrt{(k^2+\theta^2)} + k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})$

$$+2\theta]^2 + 4\theta^2 + 8k^2 - 4\}^{\frac{1}{2}}\,d\theta \quad \ldots\ldots (6).$$

Let $\theta = \frac{1}{2}k\left\{\left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^x - \left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^{-x}\right\}$

$$= \tfrac{1}{2}k[a^x - a^{-x}] = k\sinh(x\log a), \text{ where } a = \frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}$$

Hence, when $\theta=0$, $x=0$, and when $\theta=2\pi n$, $x=1$.

Then $s_1 = \frac{l}{4\pi n}\log a\int_0^1\{[\frac{1}{2}k^2\sinh 2(x\log a) + k^2\log a^x + 2k\sinh(x\log a)]^2$

$$+4k^2\sinh^2(x\log a) + 8k^2 - 4\}^{\frac{1}{2}}\cosh(x\log a)\,dx.$$

If we divide the interval (0, 1) into 10 equal parts and find the value of

$$\{[\tfrac{1}{2}k^2\sinh 2(x\log a) + k^2x\log a + 2k\sinh(x\log a)]^2$$
$$+4k^2\sinh^2(x\log a) + 8k^2 - 4\}^{\frac{1}{2}}\cosh(x\log a)$$

for each of the values of $x=0, .1, .2, \ldots\ldots, .9, 1$ and if these values be designated by $A_0, A_1, A_2, \ldots\ldots, A_{10}$, respectively, we have, by Cotes' Method of Approximate Quadrature,*

$$s_1 = \frac{l}{4\pi n}\log\left(\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right)$$

$$\times\frac{16067(A_0+A_{10}) + 106300(A_1+A_9) - 48525(A_2+A_8) + 272400(A_3+A_7)}{598752}$$

$$+\frac{-260550(A_4+A_6) + 427368A_5}{598752}.$$

*See Roger Cotes' *Opera Miscellanea*, p. 33.

The ordinates A_0, A_1, ..., A_{10} may be easily computed by means of a table of Hyperbolic Functions.

In this solution it has been assumed that the unwinding of the string begins at the vertex of the cone. If the unwinding begins at the base of the cone, we replace, in (3), θ by $(2\pi n-\theta)$ and take the negative sign of the radical since it is then a decreasing function of θ. This gives

$$s=\frac{R}{4\pi n}\Big[2\pi n\sqrt{(4\pi^2n^2+k^2)}-(2\pi n-\theta)\sqrt{[(2\pi n-\theta)^2+k^2]}$$

$$+k^2\log\Big(\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{2\pi n-\theta+\sqrt{[(2\pi n-\theta)^2+k^2]}}\Big)\Big].$$

We then have, $ds_1=[(s+\frac{(2\pi n-\theta)Rd\theta}{2\pi n})^2+ds^2+\frac{l^2-R^2}{4\pi^2n^2}d\theta^2]^{\frac{1}{2}}$ .....(7).

But $ds=-\frac{R}{2\pi n}\sqrt{[(2\pi n-\theta)^2+k^2]}d\theta$.

Substituting the values of s and ds in (7) and letting $LR/4\pi n$=the entire length of the string, and $\theta=2\pi n-k\sinh[(1-x)\log a]$, where

$$a=\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{k},$$

we have,

$$s_1=\frac{R}{4\pi n}\int_0^1\Big[\{L-\tfrac{1}{2}k^2\sinh[2(1-x)\log a]+k^2(1-x)\log a$$

$$+2k\sinh[(1-x]\log a]\}^2+4k^2\sinh^2[(1-x)\log a]$$

$$+8k^2-4\Big]^{\frac{1}{2}}\cosh[(1-x)\log a]dx,$$

the value of which may be obtained by the foregoing method of approximation.

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

Remark by DR. L. E. DICKSON, The University of Chicago.

According to A. H. Holmes (MONTHLY, 1905, p. 141), the program could be arranged in 7 evenings, A and B being partners only the first evening. But this solution is clearly erroneous, since by the last condition of the problem A and B shall play against the 15 possible pairs of the remaining six.

I proceed to prove that it is impossible to construct a program of the desired kind. The notation may be chosen so that the order of play for the first evening is

$$\text{(I)}\quad 12,\ 34,\ 56,\ 78.$$

With 12, 35, must go either 47, 68 or else 48, 67 (since 46, 78 is excluded by I). But these two cases are interchanged by the substitution (78), which does not alter I. Hence, if the problem is possible, there would be a program with I and

$$\text{(II)}\quad 12,\ 35,\ 47,\ 68.$$

Then with 12, 36 cannot go 47, 58 or 45, 78. In this way we get

$$\begin{aligned}&\text{(III)}\quad 12,\ 36,\ 48,\ 57;\\ &\text{(IV)}\quad 12,\ 37,\ 46,\ 58;\\ &\text{(V)}\quad 12,\ 38,\ 45,\ 67.\end{aligned}$$

With 13, 24 goes 57, 68 or 58, 67, cases interchanged by the substitution (5768), which leaves I unaltered and permutes II, IV, III, V, cyclically. Hence we may set

$$\begin{aligned}&\text{(VI)}\quad 13,\ 24,\ 57,\ 68;\\ &\text{(VII)}\quad 14,\ 23,\ 58,\ 67\quad\text{(by I and VI)};\\ &\text{(VIII)}\quad 16,\ 23,\ 45,\ 78\quad\text{(by III and VII)}.\end{aligned}$$

Then 15, 23 cannot go with 46, 78; 47, 68; or 48, 67, by VIII, II, VII, respectively. Hence, the problem is impossible.

First modification of problem. If we allow each couple to play exactly three times against every other couple, the problem becomes possible, there being

$$_8C_2\cdot{}_6C_2\cdot{}_4C_2\cdot{}_2C_2\div 4!=105$$

orders of the players, corresponding to the 105 substitutions of the type (12)(34)(56)(78). While this program is absolutely fair to each player, it would require 105 evenings.

Second modification. Required a program for 35 evenings of duplicate whist between eight players, such that during the series every couple shall play every other couple once and but once, while in each evening there shall be three orders of play in which no two persons play together twice.

Such a program (which is equally fair to all players) is the following:

12, 36; 58, 47	13, 28; 45, 67	12, 35; 46, 78	13, 26; 48, 57
14, 38; 25, 67	15, 27; 38, 46	14, 26; 37, 58	14, 27; 35, 68
15, 26; 34, 78	17, 25; 36, 48	18, 27; 34, 56	16, 23; 45, 78
13, 25; 47, 68	15, 24; 68, 37	12, 34; 57, 68	12, 37; 48, 56
15, 23; 48, 67	16, 28; 35, 47	13, 24; 56, 78	14, 28; 36, 57
16, 24; 38, 57	18, 23; 57, 46	17, 26; 38, 45	18, 24; 35, 67

with cyclic permutations of the last three columns;

12, 67; 35, 48	12, 45; 36, 78	12, 38; 47, 56	13, 27; 45, 68
13, 46; 25, 78	14, 56; 28, 37	17, 23; 46, 58	14, 23; 58, 67
15, 47; 23, 68	17, 68; 25, 34	18, 45; 27, 36	17, 56; 28, 34
13, 58; 26, 47	14, 78; 26, 35	15, 36; 27, 48	16, 27; 34, 58
15, 28; 34, 67	16, 25; 38, 47	16, 37; 24, 58	17, 28; 36, 45
18, 37; 25, 46	17, 58; 24, 36	17, 35; 28, 46	18, 47: 23, 56
16, 34; 27, 58	16, 58; 23, 47	16, 48; 25, 37	
17, 24; 38, 56	17, 34; 28, 56	17, 46; 23, 58	
18, 26; 37, 45	18, 25; 37, 46	18, 36; 27, 45	

This solution, which I made in 1905, has been re-checked at two different times. Note that the first 24 orders furnish a Thompson program for 24 evenings, each couple to play against the other three couples in turn. It would be interesting to know whether or not 24 is a maximum* in the Thompson problem.

134. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

How many sets of solutions has the congruence $\lambda+\mu+\nu+\xi\equiv 0 \pmod{p-1}$ p being a prime number; the order of λ, μ, ν, ξ being disregarded.

Solution by the PROPOSER.

Assume that $p>5$, and let n_i be the number of solutions in which i of the λ, μ, ν, ξ are congruent to each other (mod $p-1$). If $p\equiv 1 \pmod 4$, then $n_4=4$, viz.,

$$\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \frac{p-1}{4},\ \frac{p-1}{2},\ \frac{3(p-1)}{4}.$$

If $p\equiv 3 \pmod 4$, $n_4=2$, viz., $\lambda\equiv\mu\equiv\nu\equiv\xi\equiv 0,\ \frac{p-1}{2}$.

Next, let $i=3$, so that the congruence reduces to $\lambda+3\nu\equiv 0 \pmod{p-1}$.

*Note that a Thompson program for 22 evenings is given by (I)–(VIII) and

13, 25, 46, 78;	13, 26, 47, 58;	13, 27, 48, 56;	14, 25, 37, 68;
14, 26, 38, 57;	15, 24, 36, 78;	15, 27, 34, 68;	16, 28, 34, 57;
17, 24, 38, 56;	17, 26, 35, 48;	17, 28, 36, 45;	18, 23, 46, 57;
18, 24, 35, 67;	18, 25, 36, 47;	18, 26, 37, 45.	

These 22 orders are mutually consistent, while no other is consistent with them.

Evidently ν cannot be congruent to $0, \frac{p-1}{4}, \frac{p-1}{2}$ or $\frac{3(p-1)}{4}$, for then $\nu \equiv \lambda$. Otherwise ν can have any value (mod $p-1$), so that when $p \equiv 1 \pmod 4$, $n_3 = p-1-4 = p-5$, and when $p \equiv 3 \pmod 4$, $n_3 = p-3$.

When $i=2$, there are two possibilities.

(1) Let the parameters be congruent in pairs so that

$$2\lambda + 2\nu \equiv 0 \pmod{p-1}.$$

Here, if $\nu \equiv 0$, then must $\lambda \equiv \frac{p-1}{2}$ and inversely. Let all values (mod $p-1$) be assigned to ν in succession. With each value of ν belongs $dv(2; p-1) = 2$ values of λ determined by the congruence; as ν assumes its values λ repeats these values in another order twice. Hence, removing the two excluded solutions (0, 0), $\left(\frac{p-1}{2}, \frac{p-1}{2}\right)$, we have, if $p \equiv 1 \pmod 4$,

$$n_{2_1} = \frac{2(p-1)}{2} - 2 = p-3,$$

and if $p \equiv 3 \pmod 4$,

$$n_{2_1} = \frac{2(p-1)}{2} + 1 - 2 = p-2.$$

(2) Next suppose that two parameters are congruent and the other two incongruent, so that

$$\lambda + \mu + 2\nu \equiv 0 \pmod{p-1}.$$

For the two values $\nu \equiv 0$, $\nu \equiv \frac{p-1}{2}$ the congruence reduces to $\lambda + \mu \equiv 0 \pmod{p-1}$. Then μ is not $\equiv 0$ or $\frac{p-1}{2}$, since then $\lambda \equiv \mu$. Apart from these two exceptions μ can have any value (mod $p-1$), and for each μ a λ is determined by the congruence. Thus the cases $\nu \equiv 0, \frac{p-1}{2}$ yield $\frac{2(p-3)}{2}$ solutions. Aside from the two exceptions given ν can have any value (mod $p-1$). We may now assign to μ $p-3$ values exclusive of the value ν_1 assigned to ν, and the value $p-1-3\nu_1$. Then λ receives all values (mod $p-1$) exclusive of the two $p-1-3\nu_1$ and ν_1, respectively. Two values of λ are congruent to μ and must be excluded; *e. g.*, $\lambda \equiv \mu \equiv \frac{p-1-2\nu_1}{2}$ and $\lambda \equiv \mu \equiv p-1-\nu_1$. Aside from these, λ's values are the values of μ in another order. Hence, since $(\lambda\ \mu\ \nu)$ and $(\mu\ \lambda\ \nu)$ constitute the same solution, we obtain $(p-3)\left(\frac{p-3}{2} - 1\right) = (p-3)\left(\frac{p-5}{2}\right)$ new solutions. Thus $n_{2_2} = p-3+(p-3)\left(\frac{p-5}{2}\right) = \frac{1}{2}(p-3)^2$.

Now if we have regard to the order of λ, μ, ν, ξ, there are $(p-1)^3$ ways of satisfying the original congruence. For each of three parameters can be assigned in $p-1$ ways, and the congruence determines the remaining one. Hence we have

$$(p-1)^3 = n_4 + {}_3C_4 n_3 + 6n_{2_1} + 2_2C_4 n_{2_2} + {}_4P_4 n_1,$$
$$(p-1)^3 = n_4 + 4n_3 + 6n_{2_1} + 12n_{2_2} + 24n_1.$$

Substituting the values found above for n_{2_2}, n_{2_1}, n_3, n_4, we get, when $p \equiv 1 \pmod 4$,

$$n_1 = \tfrac{1}{24}(p^3 - 9p^2 + 29p - 21)$$

and when $p \equiv 3 \pmod 4$,

$$n_1 = \tfrac{1}{24}(p^3 - 9p^2 + 29p - 33).$$

Hence, when $p > 5$ is a prime of the form* $4l+1$ the number (N) of sets of solutions of $\lambda + \mu + \nu + \xi \equiv 0 \pmod{p-1}$ is,

$$N = \tfrac{1}{24}(p^3 - 9p^2 + 29p - 21) + \tfrac{1}{2}(p-3)^2 + p - 3 + p - 5 + 4$$
$$= \tfrac{1}{24}(p^3 + 3p^2 + 5p - 9).$$

But when p is of the form $4l+3$,

$$N = \tfrac{1}{24}(p^3 - 9p^2 + 29p - 33) + \tfrac{1}{2}(p-3)^2 + p - 2 + p - 3 + 2$$
$$= \tfrac{1}{24}(p^3 + 3p^2 + 5p + 3).$$

GEOMETRY.

284. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose that two radii R and r, whose center is the origin, revolve with uniform angular velocities 3θ and θ, respectively. What is the equation of the locus of P, the projection parallel to the X axis of the extremity of the radius r on the radius R produced if necessary.

b) Apply this curve to the trisection of an angle.

c) Suppose the ratio of their velocities is $n\theta : \theta$. Show how we can effect the multisection of an angle.

Solution by A. H. HOLMES, Brunswick, Maine.

a) Take O, the center of the circle, radius a, as the origin of coördinates. Then taking any angle θ, we shall have $r \sin 3\theta = a \sin \theta$.

$\therefore r = \dfrac{a \sin \theta}{\sin 3\theta}$ is the equation of the locus of the point P.

b) Construct the curve $r = \dfrac{a \sin \theta}{\sin 3\theta}$. On the circumference of the circle

*The values $p=3, 5$ constitute exceptions in the method employed. The final result does not hold for $p=5$. But, by inspection, when $p=5$, $N=10$; and when $p=3$, $N=3$.

take an arc equal to any angle ϕ from axis of x. Draw a line from the upper limit of the arc to the origin. From the point P, where this line cuts the curve $r=\frac{a\sin\theta}{\sin 3\theta}$, draw a line parallel to axis of x cutting the given arc. From this point draw a line to O making an angle$=\frac{1}{3}\phi$.

c) The equation of the locus of P would be $r=\frac{a\sin\theta}{\sin n\theta}$, and the multisection would be similar to the trisection.

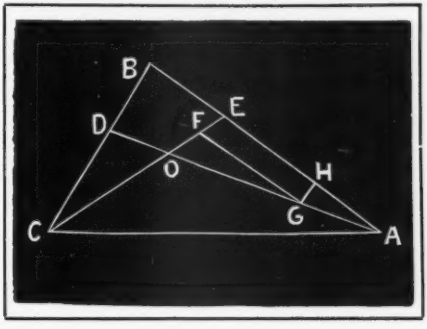
NOTE. By projecting parallel to the y-axis Dr. Zerr obtains in a similar way the locus $r=\frac{a\cos\theta}{\cos n\theta}$, and effects the n-section of the angle in a manner similar to the above. ED.

285. Proposed by G. E. BROCKWAY, Nashua, N. H.

Prove without the aid of the circle, that if the bisectors of the angles of a triangle be drawn, the greatest bisector falls on the least side.

I. Solution by ALFRED H. PARROTT, North Dakota Agricultural College, N. D.

Given scalene triangle ABC, and bisectors of angles A and C, supposing $\angle C>\angle A$. If $\angle C>\angle A$, then side $AB>$side BC, and we are to prove bisector AD on side $BC>$bisector CE on side AB. If $\angle C>\angle A$, $\angle OCA>\angle OAC$ [If wholes are unequal, then halves, etc.].



$\therefore AO>OC$ [If angles of a triangle, etc.]. Now if $OD>$ or $=OE$, the proposition is evident. But suppose $OD<OE$. Then on OE take $OF=OD$, and on OA take $OG=OC$, and draw FG. Then draw GH parallel to CE.

$\triangle FOG=\triangle COD$ [Three sides on one equal respectively, etc.]. Then $\angle FGO=\angle DCO>\angle EAO$ [Halves of unequals, etc.].

Consider $\triangle FGO$ and $\triangle EAO$; $\angle O\equiv\angle O$, $\angle FGO>\angle EAO$ [Previous proof]. Therefore $\angle GFO<\angle AEO$.

A line drawn parallel to FG and through E will then intercept HG between H and G and HG is therefore $>EF$.

Now $\angle AHG>\angle HAG$, for $\angle AHG=\angle AEC$, [sides respectively parallel], $\angle AEC=\angle B+\angle ECB$ [Exterior angle of a triangle equals sum of, etc.]. But $\angle HAG<\angle ECB$, and therefore $<\angle B+\angle ECB$, and hence $<\angle AHG$.

$\therefore AG>HG>EF$, or $AO-GO>OE-OF$,

$AO+OF>OE+GO$, $AO+OD>EO+OC$.

Hence $AD>EC$. Q. E. D.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let in a triangle ABC $AB>AC$, and BE and CD be the bisectors of angles B and C of the triangle, cutting AC and AB in E and D, respectively. To prove $CD<BE$.

Through E draw EH parallel to CB. Draw HI parallel to AC, and HK parallel to DC, cutting AC in K. It is evident that $HE=HB$, and $HE=EK$; moreover, $\angle HIB=\angle ACB>\angle HBI$. Therefore, $BH>HI$, and hence $EK>HI$ or EC. The point K, therefore, lies on AC produced, and hence, the point D between A and H. Comparing the two triangles BHE and HEK, we see at once that $BE>HK$. But $CD<HK$, *a fortiori*. Therefore $CD<BE$.

Also solved by Henry Heaton, A. H. Holmes, Rev. J. H. Meyer, and G. B. M. Zerr.

286. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

On the sides of a given triangle measure off equal distances from the extremities of the base, and at these points erect perpendiculars to the sides. Find the locus of the point of intersection of these perpendiculars. Solve by methods of analytic geometry.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ACB be the triangle, AB the base. Lay off $AE=BD=d$ on the two sides AC, BC, and erect perpendiculars at E and D cutting each other at H, and CB and AC at F and G, respectively. Then $CE=b-d$, $CF=(b-d)\sec C$.

Hence $\dfrac{x}{(b-d)\sec C}+\dfrac{y}{b-d}=1$, is the equation to EF.....(1).

Also $CD=a-d$, $CG=(a-d)\sec C$, and $\dfrac{x}{a-d}+\dfrac{y}{(a-d)\sec C}=1$, is the equation to DG.....(2). Now (1) and (2) may be written as follows:

$$x\cos C+y=b-d \text{ .....(3)},$$
$$x+y\cos C=a-d \text{ .....(4)}.$$

Eliminating d we get $x-y=\dfrac{a-b}{1-\cos C}=\frac{1}{2}(a-b)(\operatorname{cosec}\frac{1}{2}C)^2$, as the locus of H, the intersection required.

Also solved by G. W. Greenwood, Henry Heaton, A. H. Holmes, and J. Scheffer.

287. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill

Show that the points whose abscissae are 0, $a\sqrt{3}$, and $-a\sqrt{3}$ are points of inflexion on the locus $x^2y-a^2x+a^2y=0$.

Solution by the PROPOSER.

Let P be the point whose abscissa is $a\sqrt{3}$ and whose ordinate is therefore $\dfrac{a\sqrt{3}}{4}$. Let Q be any point on the curve. The coördinates of Q are, therefore,

$$a\sqrt{3}+r\cos\theta,\quad \frac{a\sqrt{3}}{4}+r\sin\theta,$$

where $PQ=r$, and the line PQ makes an angle θ with the x-axis. Hence, since Q lies on the curve, we have

$$2ar(\cos\theta+8\sin\theta)+\sqrt{3}\,ar^2\cos\theta(\cos\theta+8\sin\theta)+4r^3\cos^2\theta\sin\theta=0.$$

One value of r is zero for all values of θ; hence one branch of the curve passes through P. *Two* more values of r are zero, when $8\sin\theta+\cos\theta=0$. Hence P is a point of inflexion. In a similar manner we can show that the other points named are points of inflexion.

Also solved by A. H. Holmes, J. Scheffer, and G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Obtain the reduced cubic $4\theta^3-I\theta+J=0$ of the biquadratic $ax^4+4bx^3+6cx^2+4dx+e=0$.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$

267. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the trigonometric functions of x as infinite continued fractions.

CALCULUS.

221. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots=\frac{15}{\pi^2}.$$

222. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Evaluate $\int_0^1(1+x^m)^n\log x\,dx$.

DIOPHANTINE ANALYSIS.

137. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Prove that all multiply perfect numbers of multiplicity n having only n distinct primes are comprised in $n=2, 3, 4$.

GEOMETRY.

290. Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given nine points lying by threes in three columns and in three rows, draw through them, by continuous motion, a broken line composed of only four straight segments, and passing but once through each of the nine points. [A current puzzle.]

291. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

The pedal line of any point on a triangle's circum-circle bisects the distance between this point and the ortho-center of the triangle.

292. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Apply the locus of $(x^2+y^2)^3=mx^3$ to the problem of finding a cube m times a given cube.

293. Proposed by W. J. GREENSTREET, M. A., Editor Mathematical Gazette, Stroud, England.

A variable circle touches an ellipse, and the chord of contact through the other two points of intersection touches a similar coaxial ellipse. Find the locus of the center of the variable circle.

294. Proposed by JOHN JAMES QUINN, Ph. D., Warren, Pa.

a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

MECHANICS.

190. Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A pole hinged at the bottom leans against the mid-point of a smooth rope suspended from two supports of equal height. Determine the position of equilibrium.

192. Proposed by REV. J. H. MEYER, S. J., College of the Sacred Heart, Augusta, Ga.

Find the velocity of a planet at a given point in its orbit.

NOTES AND NEWS.

Dr. C. N. Haskins has been appointed assistant professor of mathematics in the University of Illinois.

Professor J. H. Tanner will return to his work at Cornell University this year, after a year's leave of absence.

At Cornell University Mr. W. H. Carruth has been appointed fellow, and Mr. C. F. Craig assistant in mathematics.

Professor S. E. Slocum, of the University of Illinois, has accepted the professorship of applied mathematics at the University of Cincinnati.

Mr. C. E. Colpitts, formerly assistant in mathematics at Cornell University, has been appointed adjunct professor of mathematics in the Georgia School of Technology.

Columbia University has received $5,000 with which to establish a mathematical prize, in memory of John D. Van Buren, Jr., a member of the class of 1903. The fund is the gift of Mrs. Louise T. Hoyt.

A movement is being promoted to organize a new section of the American Mathematical Society, with headquarters at St. Louis or Kansas City. It is hoped to bring into more active participation in the business of the society a large number of its members who are situated at a distance from any present section.

The following persons have been elected members of the American Mathematical Society: Rev. R. D. Carmichael, Hartselle, Ala,; Mr. F. L. Griffin, University of Chicago; Mr. W. R. Longley, University of Chicago; Mr. W. D. MacMillan, University of Chicago; Mr. F. W. Owens, Evanston Academy; Dr. J. J. Quinn, High School, Warren, Pa.; Mr. W. J. Risley, University of Illinois; Dr. R. P. Stephens, Wesleyan University; Mr. J. D. Suter, Iowa State College; Mr. A. M. Wilson, McKinley High School, St. Louis, Mo. The total membership of the society is now five hundred and thirty.

The University of Chicago announces the following advanced courses in mathematics for the summer quarter, June 19-September 1:. By Professor O. Bolza: Elliptic functions, four hours; Functions of a real valuable, four hours. By Professor H. Maschke: Geometry, four hours. By Professor H. E. Slaught: Elliptic integrals, four hours; Analytical geometry, five hours. By Professor L. E. Dickson: Algebraic analysis, four hours; Theory of substitutions, four hours. By Dr. A. C. Lunn: Integral calculus, five hours; General seminar, two hours. By Mr. N. J. Lennes: Pedagogy of mathematics, four hours.

Professor L. E. Dickson will offer, in addition to his advanced courses, a course in the correspondence study department in Plane Trigonometry, by the Laboratory Method.

Mr. William Marshall, of the University of Michigan, will spend the coming year in study abroad.

Professor G. A. Miller has been called to an associate professorship of mathematics at the University of Illinois.

Professor E. L. Richards, of Yale University, will retire from active service at the close of the present academic year.

Mr. Clarence M. Thorne has resigned his instructorship in the State University of Iowa. The position will be filled by appointment in June.

The current *Bookman* contains a contribution by Professor C. J. Keyser, of Columbia University, entitled "Concerning Research in American Universities."

Dr. Saul Epsteen, formerly instructor, has been promoted to an assistant professorship in mathematics in the University of Colorado, being now second in rank to the head of the department.

Professor Oskar Bolza, of the University of Chicago, who is spending the year in travel and study in Central Europe and Egypt, will resume his lectures at the university at the beginning of the summer quarter, June 19, 1906.

Dr. Samuel Hart Wright (M. A., M. D., Ph. D.), a versatile scholar in many fields, died at his home in Penn Yan, N. Y., on October 7, 1905, at the age of eighty years. Dr. Wright is known to the scientific world through the publication in 1848-50 of his astronomical tables, and for his studies in botany and geology. He had been a reader the MONTHLY since it was founded in 1894.

ERRATUM.

On page 85, Vol. XIII, No. 4, the term involving B_3 should have been extended as .000,000,276 instead of .000,000,009, and the final value of the integral should read 1.657,636,257 instead of 1.657,636,524.

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GROUPS OF ORDER 2^m THAT CONTAIN CYCLIC SUBGROUPS OF ORDER 2^{m-3}.

By ALICE M. McKELDEN.

INTRODUCTION.

A determination of groups of order p^3, and p^4 has been given by Hölder;* another determination of groups of orders p^3 and p^4, by Young.† The distinct types of groups of orders p^2, p^3, p^4 have been constructed and tabulated by Burnside.‡ Types of groups of order p^5, in addition to those of order p^3, p^4, which were considered first to illustrate the method of treatment, have been determined by Bagnera;§ and types of groups of order p^6 have been determined by Potron.¶ The number of groups of order p^m, which contain self-conjugate cyclic subgroups of orders p^{m-1} and p^{m-2}, respectively, has been discussed by Burnside;‖ the number of groups of order p^m that contain cyclic non-self-conjugate subgroups of order p^{m-2} has been determined by Miller;** and the groups of order p^m, which contain cyclic subgroups of order p^{m-3} (p odd prime) have been determined by Neikirk.††

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* *Mathematische Annalen*, Vol. 43 (1893), pp. 301—412.

† *American Journal of Mathematics*, Vol. 15 (1893), pp. 124—178.

‡ *Theory of Groups of a Finite Order*, pp. 81—89.

§ *Annali di Mathematica*, Vol. 3 (1898), pp. 137—228; 263—275.

¶ *These* (1904), Gauthier Villars, Paris.

‖ *Loc. Cit.* pp. 75—81. One of the groups has been omitted by Burnside and XI ~ XII, p. 81. See "A Note on Groups of Order 2^m, which contain Self-Conjugate Subgroups of Order 2^{m-2}," Hallett, *Science*, New Series, Vol. 21, No. 527, Feb. 3, 1905.

** *Transactions American Mathematical Society*, Vol. 2 (1901), p. 259, and Vol. 3 (1902), p. 383.

†† Publications of The University of Pennsylvania, Mathematics, No. 3. *Transactions of American Mathematical Society*, Vol. 6 (1905), No. 3.